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CONTINUAL MECHANICS OF MONODISPERSE SUSPENSIONS. INTEGRAL AND DIFFERENTIAL LAWS OF CONSERVATION

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A disperse medium consisting of an incompressible fluid and small spheres of equal radii suspended in the fluid is considered as the superposition of two interpenetrating and interacting continua. Equations of conservation of mass, momentum and moment of momentum are obtained for the two continua in which all unknowns are expressed in terms of functionals of mean stresses acting at the surface of an individual suspended sphere.

The mathematical definition of the motion of a disperse system – investigated in numerous works – requires the solution of two distinct problems. The first of these consists of the formal derivation of "macroscopic" equations for the system phases which are assumed to be interpenetrating continuous media with specific properties. These equations which reflect the laws of conservation of mass, momentum and moment of momentum are usually obtained by known methods of mechanics of continuous media [1, 2]. The derivation of such equations for multiphase disperse systems of various kinds is treated, for instance, in [3 – 6]. However the obtained equations contain unknown terms which have to be defined by physical parameters of phases and the unknown variables appearing in these equations, since only then can the system of equations be closed. The derivation of these rheological relationships ("equations of state") is the second object of the general problem, and requires separate analysis. The form of these relationships evidently depends on the kind of the considered disperse system.

Batchelor had shown [7] that the successful solution of the second problem requires the determination of formal relationships between the variables which define the macroscopic aspect of the flow and the quantities which determine (in the average) the hydrodynamic situation at the level of individual particles. In accordance with this the equations of conservation are derived below in a form convenient for direct formulation of relationships between macroscopic variables and mean stresses at the surface of a single suspension particle.

To simplify the problem and investigate its theoretical aspects, the monodisperse suspension is assumed to be in thermodynamic equilibrium so that it is unnecessary to take into consideration the equations of conservation of the entropy of phases, and the equations of conservation of energy become trivial corollaries of related equations of conservation of mass, momentum and moment of momentum. The flow boundaries and the interfaces of the suspension and the single-phase continuous medium, which may considerably affect the system energy balance [8], are excluded from the analysis so that the results are valid only at distances substantially greater than the mean distance between adjacent particles. Finally, we assume that the continuous suspension phase is incompressible, the particles are solid incompressible spheres, and that the Reynolds number which defines the flow around individual particles is small. According to [9] we can neglect in this case the random pulsations of particles induced by fluctuations of the suspension concentration on the shaping of its rheological properties. However the particles are not so small as to make their translational or rotational Brownian motions significant.

1. Let us consider a system of N solid spheres of radius a with their centers at points $\mathbf{r}^{(j)}$ $(j = 1, 2, \ldots, N)$ in an incompressible fluid. Such system can represent either the disperse phase of a suspension or a compact granular layer with fluid filtering through it. The number N and the total volume A occupied by the system are assumed to be large in order to make possible the application of conventional statistical methods. The problem involves the formulation of equations of conservation for the two phases of the considered model of two interpenetrating continuous media [10]. This model makes sense only if we can separate in the media such a small physical volume b in which the number of particles is sufficient for the averaging over it to be valid, but whose linear dimensions are considerably smaller than the scale of the macroscopic flow. On these assumptions it is possible to have the variables which define such flows virtually independent of coordinates within the considered volume.

The fluid macroscopic velocity and pressure fields determined in particle interstices are defined by random functions $\mathbf{V}(t, \mathbf{r})$ and $P(t, \mathbf{r})$, respectively. The translational velocity of the center of the jth particle is denoted by $\mathbf{W}^{(j)}(t)$ and its angular velocity around an axis passing through its center by $\mathbf{\Lambda}^{(j)}(t)$. it is also expedient to introduce the random vector function

$$\mathbf{W}(t, \mathbf{r}) = \begin{cases} \mathbf{W}^{(j)}(t) + \mathbf{\Lambda}^{(j)}(t) \times (\mathbf{r} - \mathbf{r}^{(j)}), & |\mathbf{r} - \mathbf{r}^{(j)}| < a \\ 0, \ |\mathbf{r} - \mathbf{r}^{(j)}| > a, & j = 1, 2, \ldots, N \end{cases} \qquad (1.1)$$

which defines the microscopic velocity of the disperse phase. In addition we introduce the microscopic velocity of the suspension

$$\mathbf{C}(t,\mathbf{r}) = \theta(t,\mathbf{r})\,\mathbf{V}(t,\mathbf{r}) + [1-\theta(t,\mathbf{r})]\,\mathbf{W}(t,\mathbf{r}) \tag{1.2}$$

which is defined at all points of volume A. In this formula $\theta(t,\mathbf{r})$ is a function which vanishes inside particles and is equal unity outside these. This function was considered in [11].

The macroscopic variables which determine the average motion of the system phases are defined by

$$\mathbf{v}(t,\mathbf{r}) = \frac{1}{\varepsilon(t,\mathbf{r})\,b}\int\limits_b \theta(t,\mathbf{r}+\mathbf{r}')\,\mathbf{V}(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}' \tag{1.3}$$

$$p(t,\mathbf{r}) = \frac{1}{\varepsilon(t,\mathbf{r})\,b}\int\limits_b \theta(t,\mathbf{r}+\mathbf{r}')\,P(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}'$$

$$\mathbf{w}(t,\mathbf{r}) = \frac{1}{\rho(t,\mathbf{r})\,b}\int\limits_b [1-\theta(t,\mathbf{r}+\mathbf{r}')]\,\mathbf{W}(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}' = \frac{1}{n(t,\mathbf{r})\,b}\sum_j \mathbf{W}^{(j)}(t)$$

$$\boldsymbol{\lambda}(t,\mathbf{r}) = \frac{1}{n(t,\mathbf{r})\,b}\sum_j \mathbf{A}^{(j)}(t), \quad \mathbf{u}(t,\mathbf{r}) = \mathbf{v}(t,\mathbf{r}) - \mathbf{w}(t,\mathbf{r})$$

$$\mathbf{c}(t,\mathbf{r}) = \frac{1}{b}\int\limits_b \mathbf{C}(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}' = \varepsilon(t,\mathbf{r})\,\mathbf{v}(t,\mathbf{r}) + \rho(t,\mathbf{r})\,\mathbf{w}(t,\mathbf{r})$$

where $\mathbf{u}(t,\mathbf{r})$ is the mean "slip" velocity of phases and $n(t,\mathbf{r})$, $\rho(t,\mathbf{r})$ and $\varepsilon(t,\mathbf{r})$ are, respectively, the denumerable (numerical) and volume concentrations of particles, and the system porosity defined by the equality

$$\varepsilon(t,\mathbf{r}) = \frac{1}{b}\int\limits_b \theta(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}', \quad \rho(t,\mathbf{r}) = \frac{4}{3}\pi a^3 n(t,\mathbf{r}) = 1 - \varepsilon(t,\mathbf{r}) \tag{1.4}$$

Vector $\mathbf{r}$ in (1.3) and (1.4) and similar expressions appearing below represents the radius vector of the center of gravity of volume b over which averaging is carried out, and the summation with respect to j is to be carried out over all particles in b.

It is further assumed that averaging over volume $b \sim l_b^3$ is equivalent to averaging over a small arbitrarily selected surface $s \sim l_b^2$, which makes it possible to determine all macroscopic variables in (1.3) and (1.4) not by volume integrals but by corresponding surface integrals. This assumption was used in all works dealing with similar investigations known to the authors.

Let us consider a volume B which satisfies the inequality $b \ll B \ll A$, using the method developed in [12], and formulate the equations for the balance of mass, momentum and moment of momentum of phases inside that volume. The related differential equations of conservation can be derived by standard methods from these integral relationships [1, 2]. We denote by S the surface bounding B and the surface of all particles inside B by S_0. The notation s and s_0 is subsequently used with the same meaning in relation to the small physical volume b. The volume and surface area of the jth particle are denoted by b_j and s_j, respectively. Surfaces S_0 and s_0 obviously represent the sums of surfaces of all spheres contained in B and b, respectively. These sums comprise the parts of surfaces of all particles cut by S or s and lying inside B or b. We also introduce symbols S_0' and s_0' to denote the surfaces of particle entirely contained inside S and s, and write $S = S_f + S_p$ and $s = s_f + s_p$, where subscripts f and p denote the parts of S and s which pass through the fluid and the particles, respectively.

2. Because of the incompressibility of the fluid and of the particle material it is possible to consider the conservation of the volume of these instead of the conservation of the mass of phases. The conditions of conservation of the volumes of fluid and suspension inside B are defined by

$$\frac{\partial}{\partial t}\int_B \theta d\mathbf{r} + \int_{S+S_0} \theta(\mathbf{V}-\mathbf{C}_S)\mathbf{n}d\mathbf{r} = 0, \quad \frac{\partial}{\partial t}\int_B d\mathbf{r} + \int_S \mathbf{C}\mathbf{n}d\mathbf{r} = 0 \qquad (2.1)$$

where n is the vector of the external normal to the surface bounding the fluid or the suspension, and the surface velocity C_S is nonzero only at S_0. Volume B and surface S can be represented in the form of sums of small physical volumes and surfaces. Equation (2.1) can, then, be integrated in two stages: first, integration over b or s and, then, summation of obtained results. Individual terms of sums for the integrals in (2.1) can be further transformed in conformity with (1.3) and (1.4), thereby expressing these in terms of macroscopic variables. These sums themselves can be considered to be integral sums and replaced by corresponding macroscopic variables. In this manner from (2.1) we obtain

$$\frac{\partial}{\partial t}\int_B \varepsilon d\mathbf{r} + \int_S \varepsilon(\mathbf{v}\mathbf{n})d\mathbf{r} = 0, \quad \int_S (\varepsilon\mathbf{v} + \rho\mathbf{w})\mathbf{n}d\mathbf{r} = 0 \qquad (2.2)$$

The condition of conservation of the disperse phase volume is obtained by subtracting the first of Eqs. (2.2) from the second, which yields

$$\frac{\partial}{\partial t}\int_B \rho d\mathbf{r} + \int_S \rho(\mathbf{w}\mathbf{n})d\mathbf{r} = 0 \qquad (2.3)$$

The first of Eqs. (2.2) and Eq. (2.3) yield the following equations of "continuity" of continua which simulate the fluid and the disperse phases:

$$\partial\varepsilon / \partial t + \nabla(\varepsilon\mathbf{v}) = 0, \quad \partial\rho / \partial t + \nabla(\rho\mathbf{w}) = 0 \qquad (2.4)$$

These equations are of the same form as the equations of conservation of volume obtained in other works (see (3 – 6, 10))

3. The conditions of conservation of the momentum of fluid and suspension inside B are of the form

$$d_0\left\{\frac{\partial}{\partial t}\int_B \theta\mathbf{V}d\mathbf{r} + \int_S \theta\mathbf{V}[(\mathbf{V}-\mathbf{C}_S)\mathbf{n}]d\mathbf{r}\right\} - \int_{S_f+S_0} \Sigma\mathbf{n}d\mathbf{r} + d_0\int_B \theta\nabla\Phi d\mathbf{r} = 0 \qquad (3.1)$$

$$\frac{\partial}{\partial t}\int_B [d_0\theta\mathbf{V} + d_1(1-\theta)\mathbf{W}]d\mathbf{r} + \int_S [d_0\theta\mathbf{V}(\mathbf{V}\mathbf{n}) + d_1(1-\theta)\mathbf{W}(\mathbf{W}\mathbf{n})]d\mathbf{r} -$$
$$\int_S \Sigma\mathbf{n}d\mathbf{r} + \int_B [d_0\theta + d_1(1-\theta)]\nabla\Phi d\mathbf{r} = 0$$

where $\Phi(t, \mathbf{r})$ is the potential of external mass forces, the same as its volume mean, d_0 and d_1 are the densities of the fluid and particle material, respectively, and $\Sigma(t, \mathbf{r})$ is the microscopic stress tensor [7]. In particle interstices the Navier-Stokes equations and

$$\Sigma(t,\mathbf{r}) = -P(t,\mathbf{r})\mathbf{I} + 2\mu_0\mathbf{E}(t,\mathbf{r}),\ \mathbf{E} = \frac{1}{2}\left\| \frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i} \right\|,\ \mathbf{I} = \|\delta_{ik}\| \quad (3.2)$$

where μ_0 is the viscosity of fluid, are valid, while inside particles the equation

$$d_1(d\mathbf{W}/dt) = \nabla\Sigma - d_1\nabla\Phi \quad (3.3)$$

is valid.

Velocities $\mathbf{V}(t,\mathbf{r})$ and $\mathbf{W}(t,\mathbf{r})$ can be presented in the form

$$\mathbf{V}(t,\mathbf{r}) = \mathbf{v}(t,\mathbf{r}) + \mathbf{V}'(t,\mathbf{r}),\quad \mathbf{W}(t,\mathbf{r}) = \mathbf{w}(t,\mathbf{r}) + \mathbf{W}'(t,\mathbf{r})$$

where vectors with primes denote fluctuations of velocities about their mean values. The moduli of these vectors are, obviously, of the same order as the modulus of slip velocity $\mathbf{u}(t,\mathbf{r})$ or of the quantity $a\lambda(t,\mathbf{r})$. Because of the assumption of smallness of the Reynolds number calculated by any of these quantities and the particle radius, the squares of these vector components must be neglected, not withstanding how low or high are the mean phase velocities $\mathbf{v}(t,\mathbf{r})$ and $\mathbf{w}(t,\mathbf{r})$. Hence, taking into account that volume means of $\mathbf{V}'(t,\mathbf{r})$ and $\mathbf{W}'(t,\mathbf{r})$ of the kind (1.3) are zero, we obtain

$$\int_S \theta\mathbf{V}(\mathbf{V}\mathbf{n})\,d\mathbf{r} = \int_S \varepsilon\mathbf{v}(\mathbf{v}\mathbf{n})\,d\mathbf{r},\quad \int_S (1-\theta)\mathbf{W}(\mathbf{W}\mathbf{n})\,d\mathbf{r} = \int_S \rho\mathbf{w}(\mathbf{w}\mathbf{n})\,d\mathbf{r}$$

The integral over the surface $S_f + S_0$ in the first of Eqs. (3.1) is equal to the sum of similar integrals over surfaces $S = S_f + S_p$, S_0' and $S_0 - S_0' - S_p$. Converting the last two integrals into volume integrals and using (3.3), we find that the integral over the volume of particles cut by S is small in comparison with the integral over the volume of all particles contained entirely inside S and can, consequently, be neglected. Thus the integral over $S_f + S_0$ in (3.1) reduces to the sum of integrals over S and S_0'.

Taking these relationships into account and transforming the integrals of microscopic quantities in (3.1) into integrals of corresponding macroscopic variables in the manner described in Sect. 2, from (3.1) we obtain

$$d_0\left[\frac{\partial}{\partial t}\int_B \varepsilon\mathbf{v}\,d\mathbf{r} + \int_S \varepsilon\mathbf{v}(\mathbf{v}\mathbf{n})\,d\mathbf{r}\right] - \int_S \sigma\mathbf{n}\,d\mathbf{r} + \int_B (\mathbf{f} + d_0\varepsilon\nabla\Phi)\,d\mathbf{r} = 0 \quad (3.4)$$

$$\frac{\partial}{\partial t}\int_B (d_0\varepsilon\mathbf{v} + d_1\rho\mathbf{w})\,d\mathbf{r} + \int_S [d_0\varepsilon\mathbf{v}(\mathbf{v}\mathbf{n}) + d_1\rho\mathbf{w}(\mathbf{w}\mathbf{n})]\,d\mathbf{r} - \int_S \sigma\mathbf{n}\,d\mathbf{r} +$$

$$\int_B (d_0\varepsilon + d_1\rho)\nabla\Phi\,d\mathbf{r} = 0$$

We have introduced here the mean stress tensor $\sigma(t,\mathbf{r})$ and the mean force $\mathbf{f}(t,\mathbf{r})$ of interaction between phases, both related to a unit of suspension volume

$$\sigma(t,\mathbf{r}) = \frac{1}{b}\int_b \Sigma(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}' \quad (3.5)$$

$$\mathbf{f}(t,\mathbf{r}) = -\frac{1}{b}\int_{S_0'} \Sigma(t,\mathbf{r}+\mathbf{r}')\,\mathbf{n}\,d\mathbf{r}' = -\frac{1}{b}\sum_j \int_{S_j} \Sigma(t,\mathbf{r}+\mathbf{r}')\,\mathbf{n}\,d\mathbf{r}' \quad (3.6)$$

Summation in (3.6) is carried out over all particles contained in b; the reason for the minus sign in this formula is that vector n is by definition directed inward particles.

From Eq. (3.4) follows the balance equation for the momentum of the disperse phase inside B

$$d_1\left[\frac{\partial}{\partial t}\int\limits_B \rho \mathbf{w} d\mathbf{r} + \int\limits_S \rho \mathbf{w}(\mathbf{w}\mathbf{n}) d\mathbf{r}\right] + \int\limits_B (-\mathbf{f} + d_1 \rho \nabla \Phi) d\mathbf{r} = 0 \qquad (3.7)$$

The first of Eqs. (3.4) and Eq. (3.7) yield the differential equation of conservation of phase momenta

$$d_0[(\partial / \partial t + \mathbf{v}\nabla)\varepsilon \mathbf{v} + \varepsilon \mathbf{v} \nabla \mathbf{v}] = \nabla \boldsymbol{\sigma} - d_0 \varepsilon \nabla \Phi - \mathbf{f} \qquad (3.8)$$

$$d_1[(\partial / \partial t + \mathbf{w}\nabla)\rho \mathbf{w} + \rho \mathbf{w} \nabla \mathbf{w}] = - d_1 \rho \nabla \Phi + \mathbf{f}$$

which with the use of (2.4) we transform into

$$d_0 \varepsilon (\partial / \partial t + \mathbf{v}\nabla) \mathbf{v} = \nabla \boldsymbol{\sigma} - d_0 \varepsilon \nabla \Phi - \mathbf{f} \qquad (3.9)$$

$$d_1 \rho (\partial / \partial t + \mathbf{w}\nabla) \mathbf{w} = - d_1 \rho \nabla \Phi + \mathbf{f}$$

Note that the macroscopic stress tensor is absent in the disperse phase. This is natural, since the neglect of chaotic pulsations of particles does not introduce any effects which could result in additional transfer of momentum by suspended particles, which could be superposed on the convective transfer by the mean motion of that phase. The effective mean stress tensor does, however, appear in the disperse phase in the presence of any significant Brownian motion of particles and, also, of pulsations induced by other causes. For instance, pulsations of particles suspended in an agitated medium lead to the appearance of stresses of the conventional Reynolds kind. Pulsations induced by fluctuations of the suspension porosity and those due to the nonlinear dependence of the interaction between phases on porosity result in the appearance of the specific tensor of "pseudo-turbulent" stresses, a phenomenon investigated in [9] at small Reynolds numbers. Pseudo-turbulent stresses are of the second order with respect to velocity $\mathbf{u}(t, \mathbf{r})$ and can be neglected because of the assumption of smallness of the Reynolds number of the flow around individual particles. In the more general case the allowance for these stresses does not present any fundamental difficulties.

4. Let us formulate the equations of balance for the moment of momentum of fluid and suspension in B

$$d_0 \frac{\partial}{\partial t}\int\limits_B \theta \mathbf{r} \times \mathbf{V} d\mathbf{r} + \int\limits_{S+S_0} \theta (\mathbf{r} \times \mathbf{V})((\mathbf{V} - \mathbf{C}_S)\mathbf{n}) d\mathbf{r} - \qquad (4.1)$$

$$\int\limits_{S_f+S_0} \mathbf{r} \times (\Sigma \mathbf{n}) d\mathbf{r} + d_0 \int\limits_B \theta \mathbf{r} \times \nabla \Phi d\mathbf{r} = 0$$

$$\frac{\partial}{\partial t}\int\limits_B [d_0 \theta \mathbf{r} \times \mathbf{V} + d_1 (1 - \theta) \mathbf{r} \times \mathbf{W}] d\mathbf{r} +$$

$$\int\limits_S [d_0 \theta (\mathbf{r} \times \mathbf{V})(\mathbf{V}\mathbf{n}) + d_1 (1 - \theta)(\mathbf{r} \times \mathbf{W})(\mathbf{W}\mathbf{n})] d\mathbf{r} -$$

$$\int\limits_S \mathbf{r} \times (\Sigma \mathbf{n}) d\mathbf{r} + \int\limits_B [d_0 \theta + d_1 (1 - \theta)] \mathbf{r} \times \nabla \Phi d\mathbf{r} = 0$$

Using the previous reasoning and the method described in [2], we obtain

$$\int_B \theta(\mathbf{r}\times\mathbf{V})\,d\mathbf{r} = \sum_k \int_{b_{(k)}} \theta(t,\mathbf{r}_k+\mathbf{r}')(\mathbf{r}_k+\mathbf{r}')\times\mathbf{V}(t,\mathbf{r}_k+\mathbf{r}')\,d\mathbf{r}' = \int_B \varepsilon(\mathbf{r}\times\mathbf{v}+\mathbf{K}_0)\,d\mathbf{r}$$

$$\int_B (1-\theta)(\mathbf{r}\times\mathbf{W})\,d\mathbf{r} = \sum_k \int_{b_{(k)}} [1-\theta(t,\mathbf{r}_k+\mathbf{r}')](\mathbf{r}_k+\mathbf{r}')\times$$

$$\mathbf{W}(t,\mathbf{r}_k+\mathbf{r}')\,d\mathbf{r}' = \int_B \rho(\mathbf{r}\times\mathbf{w}+\mathbf{K}_1)\,d\mathbf{r}$$

where $\mathbf{r}_k$ are the radius vectors of the centers of gravity of volumes $b_{(k)}$ into which volume B is divided. Here we again use the definitions (1.3) and (1.4) and introduce the quantities

$$\mathbf{K}_0(t,\mathbf{r}) = \frac{1}{\varepsilon(t,\mathbf{r})\,b}\int_b \theta(t,\mathbf{r}+\mathbf{r}')\,\mathbf{r}'\times\mathbf{V}(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}' \tag{4.2}$$

$$\mathbf{K}_1(t,\mathbf{r}) = \frac{1}{\rho(t,\mathbf{r})\,b}\int_b [1-\theta(t,\mathbf{r}+\mathbf{r}')]\,\mathbf{r}'\times\mathbf{W}(t,\mathbf{r}+\mathbf{r}')\,d\mathbf{r}' =$$

$$\frac{1}{n(t,\mathbf{r})\,b}\sum_j(\mathbf{r}^{(j)}-\mathbf{r})\times\mathbf{W}^{(j)}(t) + \frac{J}{\rho(t,\mathbf{r})\,b}\sum_j \mathbf{\Lambda}^{(j)}(t) = \frac{2}{5}a^2\boldsymbol{\lambda}(t,\mathbf{r})$$

where J is the moment of inertia of a sphere of unit density and radius a about an axis passing through its center. In transforming the second formula in (4.2) we made use of the definition (1.1) and of the fact that the translation velocity of a particle contained in a volume of the order of b can be considered to be a random vector quantity whose mean properties are independent of the specific position of a particle in b. Taking into account that summation of vector quantities $\mathbf{r}^{(j)} - \mathbf{r}_k$ over all spheres in $b_{(k)}$ yields zero, we obtain the last expression in (4.2).

The quantity $d_0\mathbf{K}_0(t, \mathbf{r})$ the inner moment of momentum of the continuum which simulates the fluid phase of suspension, and is introduced in accordance with the general method described in [2]. The appearance of this moment is due to vortices of microscopic fluid motion, which vanish in the process of averaging over the volume and, consequently, do not affect the mean velocity of fluid. The inner moment of momentum of the continuum simulating the disperse phase is produced by the rotation of particles and is equal $d_1\mathbf{K}_1(t, \mathbf{r})$. The inner moments of momenta appearing in (4.2) relate to unit volumes of respective phases. The inner moments of momenta related to a unit volume of suspension are $d_0\varepsilon\mathbf{K}_0$ and $d_1\rho\mathbf{K}_1$ Similarly to $\mathbf{v}(t, \mathbf{r})$, $\mathbf{w}(t, \mathbf{r})$, etc. these quantities define the macroscopic motion of the considered interpenetrating and interacting continuous media. Taking in addition into account the smallness of fluctuations of microscopic phase velocities, we can readily transform the second integrals in formulas (4.1) in a similar manner.

Using (3.5) and (3.6), we further obtain

$$\int_{S_0} \mathbf{r}\times(\Sigma\mathbf{n})\,d\mathbf{r} = \sum_k \int_{s_{0,k}'} (\mathbf{r}_k+\mathbf{r}')\times(\Sigma\mathbf{n})\,d\mathbf{r} = \int_B \mathbf{r}\times\mathbf{f}\,d\mathbf{r} + \sum_k \int_{s_{0,k}'} \mathbf{r}'\times(\Sigma\mathbf{n})\,d\mathbf{r}'$$

$$\int_{s_{0,k}'} \mathbf{r}'\times(\Sigma\mathbf{n})\,d\mathbf{r}' = \sum_j(\mathbf{r}^{(j)}-\mathbf{r}_k)\times\int_{s_j}\Sigma\mathbf{n}\,d\mathbf{r} + \sum_j\int_{s_j}(\mathbf{r}-\mathbf{r}^{(j)})\times(\Sigma\mathbf{n})\,d\mathbf{r}$$

where s_{0k}' are the surfaces of all particles lying entirely inside volumes $b_{(k)}$. The first term in the right-hand side of the last equation vanishes for the same reasons as the structurally similar term in the second of Eqs. (4.2). This is so because the mean properties of the random force acting on a sphere are independent of the position of the radius vector of the sphere center, since that vector can only vary within the boundaries of volume of the order of b. The summation of vector quantities $\mathbf{r}^{(j)} - \mathbf{r}_k$ over all spheres in $b_{(k)}$ yields, as previously, zero.

Taking the above into consideration, from (4.1) we obtain

$$d_0\left[\frac{\partial}{\partial t}\int_B \varepsilon(\mathbf{r}\times\mathbf{v}+\mathbf{K}_0)\,d\mathbf{r}+\int_S \varepsilon(\mathbf{r}\times\mathbf{v}+\mathbf{K}_0)(\mathbf{vn})\,d\mathbf{r}\right]-\int_S \chi^*\mathbf{n}\,d\mathbf{r}+ \tag{4.3}$$
$$\int_B [\mathbf{r}\times(d_0\varepsilon\Delta\Phi+\mathbf{f})+\mathbf{m}]\,d\mathbf{r}=0$$

$$\frac{\partial}{\partial t}\int_B [d_0\varepsilon(\mathbf{r}\times\mathbf{v}+\mathbf{K}_0)+d_1\rho(\mathbf{r}\times\mathbf{w}+\mathbf{K}_1)]\,d\mathbf{r}+$$
$$\int_S [d_0\varepsilon(\mathbf{r}\times\mathbf{v}+\mathbf{K}_0)(\mathbf{vn})+d_1\rho(\mathbf{r}\times\mathbf{w}+\mathbf{K}_1)(\mathbf{wn})]\,d\mathbf{r}-$$
$$\int_S \chi^*\mathbf{n}\,d\mathbf{r}+\int_B (d_0\varepsilon+d_1\rho)\,\mathbf{r}\times\nabla\Phi\,d\mathbf{r}=0$$

We have introduced here the mean tensor of "moment" stresses $\chi^*(t, \mathbf{r})$ and the mean moment $\mathbf{m}(t, \mathbf{r})$, of interaction between phases, both related to a unit volume of suspension

$$\chi_{im}^*(t, \mathbf{r})=\frac{1}{b}\int_b \varepsilon_{ink}(r_n+r_n')\,\Sigma_{km}(t, \mathbf{r}+\mathbf{r}')\,d\mathbf{r}' \tag{4.4}$$

$$\mathbf{m}(t, \mathbf{r})=-\frac{1}{b}\sum_j(\mathbf{r}+\mathbf{r}'-\mathbf{r}^{(j)})\times(\Sigma(t, \mathbf{r}+\mathbf{r}')\,\mathbf{n})\,d\mathbf{r}' \tag{4.5}$$

where ε_{ink} is the alternating antisymmetric Levi-Civita tensor. From (4.3) also follows the condition of balance of the moment of momentum of the disperse phase

$$d_1\left[\frac{\partial}{\partial t}\int_B \rho(\mathbf{r}\times\mathbf{w}+\mathbf{K}_1)\,d\mathbf{r}+\int_S \rho(\mathbf{r}\times\mathbf{w}+\mathbf{K}_1)(\mathbf{wn})\,d\mathbf{r}\right]+ \tag{4.6}$$
$$\int_B [\mathbf{r}\times(d_1\rho\nabla\Phi-\mathbf{f})-\mathbf{m}]\,d\mathbf{r}=0$$

The first of Eqs. (4.3) and Eq. (4.6) with allowance for (2.4) yield

$$d_0\varepsilon(\partial/\partial t+\mathbf{v}\nabla)(\mathbf{r}\times\mathbf{v}+\mathbf{K}_0)=\nabla\chi^*-\mathbf{r}\times(d_0\,\varepsilon\,\nabla\Phi+\mathbf{f})-\mathbf{m} \tag{4.7}$$
$$d_1\rho(\partial/\partial t+\mathbf{w}\nabla)(\mathbf{r}\times\mathbf{w}+\mathbf{K}_1)=-\mathbf{r}\times(d_1\rho\nabla\Phi-\mathbf{f})+\mathbf{m}$$

Taking into account

$$\frac{\partial\chi_{ij}^*}{\partial r_j}-\varepsilon_{ink}r_n\frac{\partial\sigma_{kj}}{\partial r_j}=\frac{\partial}{\partial r_j}(\chi_{ij}^*-\varepsilon_{ink}r_n\sigma_{kj})+\varepsilon_{ink}\sigma_{kn}^{(a)}$$

where $\sigma^{(a)}$ is the virtual antisymmetric part of the mean stress tensor, and substituting for (4.4) the new tensor of mean moment stresses

$$\chi_{ij}(t, \mathbf{r})=\chi_{ij}^*(t, \mathbf{r})-\varepsilon_{ink}r_n\sigma_{kj}(t, \mathbf{r})=\frac{1}{b}\int_b \varepsilon_{ink}r_n'\Sigma_{kj}(t, \mathbf{r}+\mathbf{r}')\,d\mathbf{r}' \tag{4.8}$$

after subtracting the vector products of $\mathbf{r}$ by the equations of conservation of momentum (3.9) from the corresponding equations (4.7), we obtain

$$d_0\varepsilon(\partial/\partial t+\mathbf{v}\nabla)\mathbf{K}_0=\nabla\chi-\mathbf{h}-\mathbf{m},\quad h=\|\varepsilon_{ink}\sigma_{nk}^{(a)}\| \qquad (4.9)$$
$$d_1\rho(\partial/\partial t+\mathbf{w}\nabla)\mathbf{K}_1=\mathbf{m},\quad \nabla\chi=\|\partial\chi_{ij}/\partial r_j\|$$

These equations define the conservation of the moment of momentum of the considered continua. Besides tensor (4.8) we have in these equations the mean moment of the force of interaction between phases (4.5) and the additional volume moment $\mathbf{h}(t, \mathbf{r})$ produced by the antisymmetric component of tensor (3.5), as determined by (4.9). Note that quantities of similar meaning were used earlier in numerous works, in [13, 14] in particular, on general covariant considerations.

Note that the equation of inner moment of momentum of the continuum simulating the disperse phase can be derived entirely independently. Using the equation

$$d_1\mathbf{J}(d\mathbf{\Lambda}^{(j)}/dt)=\mathbf{M}^{(j)} \qquad (4.10)$$

for defining the rotation of a single sphere, where $\mathbf{M}^{(j)}$ is the moment of forces acting on that sphere, and averaging over all spheres inside the small physical volume in conformity with formulas (1.3), after multiplication by the denumerable concentration of particles $n(t, \mathbf{r})$, we obtain again the second of Eqs. (4.9) which determines the inner moment of momentum $\mathbf{K}_1(t, \mathbf{r})$ or the mean angular velocity $\boldsymbol{\lambda}(t, \mathbf{r})$.

We thus have a system of two scalar equations (2.4) and four independent vector equations (3.9) and (4.9). This system is open, since the expressions for vectors of mean force and of the moment of the force of interaction between phases and for tensors of mean and moment stresses are unknown. These quantities are considered in more detail in Sect. 6, where it is shown that they can be expressed in terms of scalar unknowns $\rho(t, \mathbf{r})$ and $p(t, \mathbf{r})$, and of functionals of mean stresses at the surface of an individual sphere. Considering that such functionals are usually represented in terms of kinematic characteristics of the mean (macroscopic) phase motion (see, e.g. [7]), we conclude that this system must serve for the determination of the two unknown scalars $\rho(t, \mathbf{r})$ and $p(t, \mathbf{r})$ and four unknown vectors $\mathbf{v}(t, \mathbf{r})$, $\mathbf{w}(t, \mathbf{r})$, $\mathbf{K}_0(t, \mathbf{r})$ and $\mathbf{K}_1(t, \mathbf{r})$ (or $\boldsymbol{\lambda}(t, \mathbf{r})$).

5. Let us write down separately the equations of conservation for two particular limit cases, viz: the motion of fluid in a stationary granular layer of specific properties, and the "homogeneous" model of a suspension in which mean velocities of the two phases may be considered to be approximately equal and the same as the mean velocity $\mathbf{c}(t, \mathbf{r})$ of the suspension.

In the first case from the results of Sects. 2 – 4 we obtain the modified Darcy's equations which define the filtration of a fluid through a layer of known porosity

$$\nabla(\varepsilon\mathbf{v})+0,\quad d_0\varepsilon(d/\partial t+\mathbf{v}\nabla)\mathbf{v}=\nabla\sigma-d_0\varepsilon\nabla\Phi-\mathbf{f} \qquad (5.1)$$
$$d_0\varepsilon(\partial/\partial t+\mathbf{v}\nabla)\mathbf{K}_0=\nabla\chi-\mathbf{m}-\mathbf{h}$$

The equations are of particular interest in connection with the continued discussion about the correct presentation of the viscous and inertial terms in equations of the theory of filtration (see e.g. [15]).

In the second case, assuming that $\mathbf{v}(t, \mathbf{r})\approx\mathbf{w}(t, \mathbf{r})\approx\mathbf{c}(t, \mathbf{r})$ and summating pairwise the equations of conservation of mass, momentum and moment of momentum of phases for the determination of motion, we have the following equations:

$$(\partial/\partial t)\, d + \nabla(d\mathbf{c}) = 0,\ d\,(\partial/\partial t + \mathbf{c}\nabla)\,\mathbf{c} = \nabla\sigma - d\nabla\Phi \tag{5.2}$$
$$d\,(\partial/\partial t - \mathbf{c}\nabla)\,\mathbf{K} = \nabla\chi - \mathbf{h}$$
$$(d\mathbf{K} = \varepsilon d_0\mathbf{K}_0 + \rho d_1\mathbf{K}_1,\ \ d = \varepsilon d_0 + \rho d_1)$$

If $\sigma(t, \mathbf{r})$ is expressed in terms of unknown variables, the mean pressure term $p(t, \mathbf{r})$ appears in (5.2) (see, e.g. formulas (6.6) below), hence system (5.2) is subdefinite, since we have one scalar and two vector equations for the determination of two unknown scalars $\rho(t, \mathbf{r})$ and $p(t, \mathbf{r})$ and two vectors $\mathbf{c}(t, \mathbf{r})$ and $\mathbf{K}(t, \mathbf{r})$. This shows the intrinsic inconsistency of the single-velocity homogeneous model of suspension.

6. According to [7] the initial step in the derivation of expressions for tensors of mean and moment stresses and, also, of mean forces and of moment of interphase interaction force in terms of specific functions (or functionals) of physical parameters defining the fluid and the suspension particles and of unknown variables of equations of conservation must be the determination of expressions for these quantities in the form of functionals of stresses at the surface of an individual particle of the suspension.

Formulas (3.6) and (4.5) for the force and moment of interaction can be rewritten in the form

$$\mathbf{f} = n\langle\mathbf{F}\rangle,\quad \mathbf{m} = n\langle\mathbf{M}\rangle,\quad \langle\mathbf{F}\rangle = \Big\langle\int_{s_j}\Sigma\mathbf{n}\,d\mathbf{r}\Big\rangle \tag{6.1}$$
$$\langle\mathbf{M}\rangle = \Big\langle\int_{s_j}(\mathbf{r} - \mathbf{r}^{(j)})\times(\Sigma\mathbf{n})\,d\mathbf{r}\Big\rangle$$

where $\langle\mathbf{F}\rangle$ and $\langle\mathbf{M}\rangle$ are, respectively, the force and the moment exerted by the fluid on an individual particle and averaged over particles contained in the small physical volume with its center of gravity at some point $\mathbf{r}$. Unlike in Sects. 2 – 4, we use in (6.1) and subsequently the vector $\mathbf{n}$ of the unit normal external to the particle and not to the surrounding fluid.

We transform the expression for $\sigma(t, \mathbf{r})$ in (3.5) in accordance with [7] and obtain

$$\sigma_{ik}(t, \mathbf{r}) = \frac{1}{b}\int_{b-b_0}\Big[-P\delta_{ik} + \mu_0\Big(\frac{\partial V_i}{\partial r_k} + \frac{\partial V_k}{\partial r_i}\Big)\Big]_{t,\,\mathbf{r}+\mathbf{r}'} dr' + \tag{6.2}$$
$$\frac{1}{b}\int_{b_0}\Sigma_{ik}(t, \mathbf{r}+\mathbf{r}')\,d\mathbf{r}'$$

where b_0 is the volume of particles contained in b and $\Sigma(t, \mathbf{r})$ is as defined by (3.2). Using the obvious relationships

$$\frac{1}{b}\int_{b-b_0}\frac{\partial V_i}{\partial r_k}\,d\mathbf{r}' = \frac{\partial c_i}{\partial r_k} - \frac{1}{b}\int_{s_0}C_{S,i}\,n_k\,d\mathbf{r}'$$
$$\int_{b_0}\Sigma_{ik}\,d\mathbf{r}' = \int_{s_0}\Sigma_{im}r_k'n_m\,d\mathbf{r}' - \int_{b_0}\frac{\partial\Sigma_{im}}{\partial r_m}r_k'\,d\mathbf{r}'$$
$$\frac{\partial c_i}{\partial r_k} = \frac{\partial}{\partial r_k}\Big\{\frac{1}{b}\int_b C_i\,d\mathbf{r}'\Big\} = \frac{1}{b}\int_b\frac{\partial C_i}{\partial r_k}\,d\mathbf{r}'$$

and taking into account (1.3) and (3.3), we reduce (6.2) to the form

$$\sigma_{ik} = -\varepsilon p\delta_{ik} + 2\mu_0 e_{ik} + \frac{1}{b}\sum_j \int_{s_j} [\Sigma_{im} r_k' n_m - \mu_0 (C_{s,i} n_k + C_{S,k} n_i)] d\mathbf{r}' - \tag{6.3}$$
$$\frac{d_1}{b}\sum_j \int_{b_j} \left(\frac{dW_i}{dt} + \frac{\partial\Phi}{\partial r_i}\right) r_k' \, d\mathbf{r}'$$

where

$$\mathbf{e} = \frac{1}{2}\left\| \frac{\partial c_i}{\partial r_k} + \frac{\partial c_k}{\partial r_i} \right\| \tag{6.4}$$

is the mean strain rate tensor.

Note that the second term of the surface integral in (6.3) vanishes owing to the impermeability and incompressibility of particles. Other integrals appearing in (6.3) can be tranformed as follows:

$$\int_{s_j} \Sigma_{im} r_k' n_m \, d\mathbf{r}' = (r_k^{(j)} - r_k) \int_{s_j} \Sigma_{im} n_m \, d\mathbf{r} + \int_{s_j} \Sigma_{im} (r_k - r_k^{(j)}) n_m \, d\mathbf{r}$$

$$\int_{b_j} \left(\frac{dW_i}{dt} + \frac{\partial\Phi}{\partial r_i}\right) r_k' \, d\mathbf{r}' = (r_k^{(j)} - r_k) \int_{b_j} \left(\frac{dW_i}{dt} + \frac{\partial\Phi}{\partial r_i}\right) d\mathbf{r} +$$
$$\int_{b_j} \left(\frac{dW_i}{dt} + \frac{\partial\Phi}{\partial r_i}\right) (r_k - r_k^{(j)}) n_m \, d\mathbf{r}$$

where $\mathbf{r}$ is the radius vector of the center of gravity of volume b in (6.3). The first two terms in the right-hand sides of these equalities obviously do not affect (6.3), since they vanish in the summation over all spheres contained in b. In fact, owing to the independence of the force acting on an individual particle in b and of the quantity $dW_i / dt + \partial\Phi / \partial r_i$ from coordinates within the boundaries of volume b, these sums reduce to the sum of quantities $r_k^{(j)} - r_k$.

Using (1.1) and (4.10), we further obtain

$$\int_{b_j} \left(\frac{dW_i}{dt} + \frac{\partial\Phi}{\partial r_i}\right) (r_k - r_k^{(j)}) \, d\mathbf{r} = \int_{b_j} \varepsilon_{imn} \frac{d\Lambda_m^{(j)}}{dt} (r_n - r_n^{(j)}) (r_k - r_k^{(j)}) \, d\mathbf{r} =$$
$$\frac{1}{2} I \varepsilon_{imk} \frac{d\Lambda_m^{(j)}}{dt} = \frac{1}{2d_1} \varepsilon_{imk} M_m^{(j)}$$

hence the expressions for the symmetric and antisymmetric components of tensor (6.3) are of the form

$$\sigma_{ik}^{(s)} = -\varepsilon p\delta_{ik} + 2\mu_0 e_{ik} + \frac{1}{b}\sum_j \overset{(s)}{\int_{s_j}} \Sigma_{im} (r_k - r_k^{(j)}) n_m \, d\mathbf{r} \tag{6.5}$$
$$\sigma_{ik}^{(a)} = \frac{1}{b}\sum_j \left(-\frac{1}{2}\varepsilon_{imk} M_m^{(j)} + \overset{(a)}{\int_{s_j}} \Sigma_{im} (r_k - r_k^{(j)}) n_m \, d\mathbf{r}\right)$$

Allowing for (4.5) and multiplying the expression for $\sigma_{ik}^{(a)}$ in (6.5) by ε_{nki}, we conclude that the antisymmetric part of the mean stress tensor and, consequently, also the volume moment $\mathbf{h}(t, \mathbf{r})$ in (4.9), (5.1) and (5.2) vanish. As the result we have

$$\sigma = -\varepsilon p\mathbf{I} + 2\mu_0\mathbf{e} + n\langle\mathbf{Y}\rangle, \quad \langle Y\rangle_{ik} = \left\langle \int\limits_{s_j}^{(s)} \Sigma_{im}(r_k - r_k^{(j)})\, n_m\, d\mathbf{r} \right\rangle \tag{6.6}$$

The structure of this expression is similar to that in (6.1). Similarly, the expression (4.8) for the tensor of mean moment stresses can be rewritten as

$$\chi_{ik} = \frac{1}{b}\int\limits_{b-b_j} \varepsilon_{imn} r_m' \left[-P\delta_{nk} + \mu_0\left(\frac{\partial V_n}{\partial r_k} + \frac{\partial V_k}{\partial r_n}\right)\right]_{t,\,\mathbf{r}+\mathbf{r}'} d\mathbf{r}' + \tag{6.7}$$

$$\frac{1}{b}\sum_j \int\limits_{b_j} \varepsilon_{imn} r_m' \Sigma_{nk}(t, \mathbf{r}+\mathbf{r}')\, d\mathbf{r}'$$

This expression can be further transformed similarly to the transformation of the expression for the mean stress tensor (6.2). Omitting intermediate computations, we present the final formula (6.8)

$$\chi = n\langle\mathbf{X}\rangle, \quad \langle X\rangle_{ik} = \left\langle \int\limits_{s_j} \varepsilon_{imn}(r_m - r_m^{(j)})(r_k - r_k^{(j)})\,\Sigma_{nq} n_q\, d\mathbf{r} \right\rangle - \frac{a^2}{5n}\varepsilon_{ikj} f_j$$

which follows from (6.7) and has the same structure as (6.1) and (6.6). We stress that $\mathbf{m}(t, \mathbf{r})$ and $\chi(t, \mathbf{r})$ are not true tensor quantities but represent a pseudo-vector and a pseudo-tensor, respectively.

The above analysis shows that the tensor of mean stress in a suspension must be symmetric. Hence the a priori introduction of antisymmetric stresses in the equations of conservation for suspensions, as was done on phenomenological considerations, may not have any physical meaning. However, it follows from general physical considerations, as shown in [7], antisymmetric tensors appear in the definition of flow of a particle suspension in the presence of an external field of nonzero dipole moments, which results in each particle being subjected to a force couple.

Because of the evident commutativity of averaging operations over spheres in a small physical volume and integration over the surface of an individual sphere, the expressions for quantities $\langle\mathbf{F}\rangle$, $\langle\mathbf{M}\rangle$, $\langle\mathbf{Y}\rangle$ and $\langle\mathbf{X}\rangle$ can be written in the form

$$\langle\mathbf{F}\rangle = \int\limits_{s_j} \langle\Sigma\rangle\, \mathbf{n}\, d\mathbf{r}, \quad \langle\mathbf{M}\rangle = \int\limits_{s_j} (\mathbf{r} - \mathbf{r}^{(j)}) \times (\langle\Sigma\rangle\, \mathbf{n})\, d\mathbf{r} \tag{6.9}$$

$$\langle Y\rangle_{ik} = \int\limits_{s_j}^{(s)} \langle\Sigma\rangle_{im}(r_k - r_k^{(j)})\, n_m\, d\mathbf{r},$$

$$\langle X\rangle_{ik} = \int\limits_{s_j} \varepsilon_{imn}(r_m - r_m^{(j)})(r_k - r_k^{(j)})\,\langle\Sigma\rangle_{nq} n_q\, d\mathbf{r}$$

so that the closing of the obtained system of equations of conservation reduces to the problem of determination of mean stresses at the surface of a single particle. This problem can be solved by methods described in [16].

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